Spiking the Volatility Punch

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1 An Introduction to VIX and SPIKES

The primary volatility index quoted in the financial press has been VIX, since it was first created in 1993. The construction of VIX was changed in 2003 to take advantage of conceptual breakthroughs in theoretical replicating strategies for over-the-counter variance swaps. Since its re-design in 2003, VIX^2 has been the cost of a portfolio of out-of the money (OTM) options written on the S&P 500. The weighting scheme combines the mid-point of bid and ask SPX option quotes at almost all of the OTM strikes and for two maturities which straddle 30 days. The across-strike weights are designed to equate the cash gamma contribution from each option, while the across-maturity weights are designed to target a 30 day forecasting horizon. The option quotes used are updated very minute. The emphasis on bid and ask quotes rather than on trades has lead prominent academics to question whether VIX can be manipulated, see [5]. The excessive volumes in out-of-the-money SPX options on VIX settlement were found to have no other explanation.

In response to the controversy surrounding possible VIX manipulation, an alternative volatility index called SPIKES has been recently introduced. In this paper, we introduce the SPIKES volatility index, primarily by comparing it to VIX, which is much more well known. Like VIX, SPIKES seeks to forecast S&P 500 volatility over a 30 day horizon. SPIKES uses the same weighting scheme across strikes and maturities as VIX, but differs from VIX in several ways:

- Dividend Timing Difference: SPIKES is calculated from prices of OTM options written on the SPY exchange traded fund (ETF), rather than from OTM SPX option prices. The underlying SPY ETF pays dividends quarterly, whereas the 500 stocks comprising S&P500 pay dividends much more frequently.
- Early Exercise Premium: SPY options have an American-style exercise feature, whereas SPX options, have a European-style exercise feature.

- Trade Prioritization: SPIKES uses a price-dragging technique to select option prices in the portfolio which gives priority to trades. This leads to greater stability over time in the option prices and hence to the volatility index.
- Weight Updating: The weights on OTM option prices in SPIKES are updated on the order of seconds, rather than minutes.

In this paper, we focus on the difference between SPIKES and VIX that arises just from differences in dividend timing and in exercise styles. We decompose the difference between SPIKES and VIX into the sum of a dividend timing difference (DTD) for European options and from the early exercise premium (EEP) for options on an ETF paying dividends quarterly. Due to the inclusion of both puts and calls in the two volatility indices, the DTD for European options can have either sign even in months where no quarterly dividend is paid. In contrast, under positive interest rates, the EEP under quarterly dividends is always positive, even in the 8 months of the year where no quarterly dividends is paid. Under the hypothetical case of zero interest rate, the EEP vanishes in the 8 months of the year when no quarterly dividends are paid and becomes positive again in the other 4 months.

On average, VIX is an upward biased forecast of subsequent realized volatility. Since the DTD can either add to or subtract from the non-negative EEP, it is not obvious ex ante whether or not SPIKES has higher upward bias than VIX. Historically, the difference between SPIKES and VIX has been positive, but small in magnitude. One can use an option valuation model to gauge how large this difference can become as US dividend yields and interest rates rise from their current positive but historically low levels.

In a hypothetical case of all 500 stocks in the S&P500 paying zero dividends, the DTD is clearly zero, while the EEP is positive under positive interest rates. An increase in dividend yields creates a non-zero DTD which again can have either sign. In standard valuation models such as Black Merton Scholes (BMS), the American call EEP increases with dividends and decreases with interest rates, while the American put EEP has the opposite behavior in dividends and interest rates. As a result, an increase or decrease in dividends has a mixed effect on the overall EEP across calls and puts, as does an increase or decrease in interest rates. The main objective of this paper is to use the benchmark Black Scholes model to calculate the magnitudes of the DTD and the EEP, and thereby to assess the magnitude of future differences between SPIKES and VIX.

We begin with a data analysis in the next section which shows that the two volatility indices have behaved very similarly in the past. We therefore turn to the harder question of whether they are likely to behave similarly going forward. The main objective of this paper is to use the benchmark BMS model to assess whether SPIKES is likely to continue tracking VIX going forward. We first examine the simpler case when the SPY ETF is not paying a quarterly divided before the SPY options mature. We then address the determination of the DTD and the EEP in the BMS model when the underlying SPY ETF pays a single proportional dividend before maturity. We also examine the behavior of the early exercise boundary and American option vegas in the BMS model with and without a quarterly dividend. Our numerical results indicate that the difference is likely to remain small so long as 30 day interest rates and annualized dividend yields both remain below 10 % per year.

2 Data Analysis from 2005 to 2018

An analysis of data from 2005 to 2018 shows that the inclusion of the positive EEP in the SPIKES calculation has had a negligible impact on the average SPIKES level. The average values of SPIKES and VIX have hardly differed during this sample period. The simplest explanation for this negligible difference is that all of the options used in the volatility index calculations are OTM, and hence their EEP is smaller than if ITM option prices were used. It also helps that US interest rates have averaged lower in this period than they did in the late 1970's, when inflation was larger. It is possible that a return of higher interest rates or a sharp increase in dividend payouts would increase the gap between SPIKES and VIX. One of the primary purposes of this paper is to use the benchmark Black Scholes model to assess the potential impact of a rise in interest rates or dividend yields on the gap between SPIKES and VIX. One can also consider the use of an alternative American option pricing model that is consistent with the volatility skew. We thought it prudent to begin with the perhaps overly simple but familiar Black Scholes model. We could then use the results of this preliminary investigation to then assess whether a more complicated but more realistic option valuation model would change our qualitative conclusions.

Table 1 indicates that the mean level of SPIKES at 18.9 has been slightly higher than that of VIX at 18.7. This gap is probably due to the EEP, but one should also recall that SPX and SPY have different dividend payout frequencies. The two volatility indices also have very similar standard deviations and skewness. The percentage changes in the two volatility indices are virtually indistinguishable over daily, weekly, and monthly horizons. The two volatility indices have had very similar, strongly negative correlations to the S&P 500 over the 3 different horizons. Moreover, the correlation between both levels and percentage changes in the two volatility indices is very close to 1. In different S&P 500 return regimes, the two volatility indices have had nearly identical behavior on average.

As reported in Table 2, if SPIKES log-differences are regressed on VIX log-differences with no intercept, then the estimated slope as well as the $\mathbb{R}^2$ of the regression are both very close to 1. This result holds for daily, weekly, and monthly horizons. The table also shows the result of regressing SPY on SPX after adjusting for the different dividend payout times:

In Figure 1, the regression fit of SPIKES returns on VIX returns is plotted for weekly and

Statistics	Index	Daily Ret.	Weekly Ret.	Monthly Ret.
Vol. Index	VIX, SPIKES	VIX, SPIKES	VIX, SPIKES	VIX, SPIKES
Mean	18.7, 18.9	0.28%, 0.28%	1.1%, 1.1%	3.1%, 3.0%
Std. Dev.	9.2, 9.2	123%, 123%	115%, 112%	96%, 94%
Skewness	2.5, 2.5	2.2, 2.9	2.8, 2.8	2.7, 2.9
SPX Corr.	-51% , -50%	-72%, -71%	-71%, -71%	-69%, -69%
VIX Corr.	1, 99.9%, 1, 97.4%,	1, 98.5%	1, 99.1%	

Table 1: Descriptive statistics for VIX, SPIKES and their returns from 2005 to 2018

Asset Pair	Daily Return R^2	Weekly Return R^2	Monthly Return R^2
SPIKES vs. VIX	0.95	0.97	0.98
SPY vs. SPX	0.98	0.99	1.00

Table 2: The R^2 value when SPIKES returns are regressed on VIX returns (top row) and SPY returns are regressed on SPX returns (bottom row).

monthly log-differences.

In Figure 2, the percentage daily difference is plotted. It is clear from the picture that SPIKES is higher than VIX especially around quarters every year, corresponding to SPY ex-dividend dates.

Figure 3 examines the difference between SPIKES and VIX by calendar month; the difference is highest in the months which contain SPY ex-dividend dates (February, May, August and November). The early exercise premium is largest one week before expiry.

3 Computing SPIKES

The weighting scheme for both SPIKES and VIX is based on a theoretical result for replicating the payoff on a variance swap via a static position in OTM European-style index

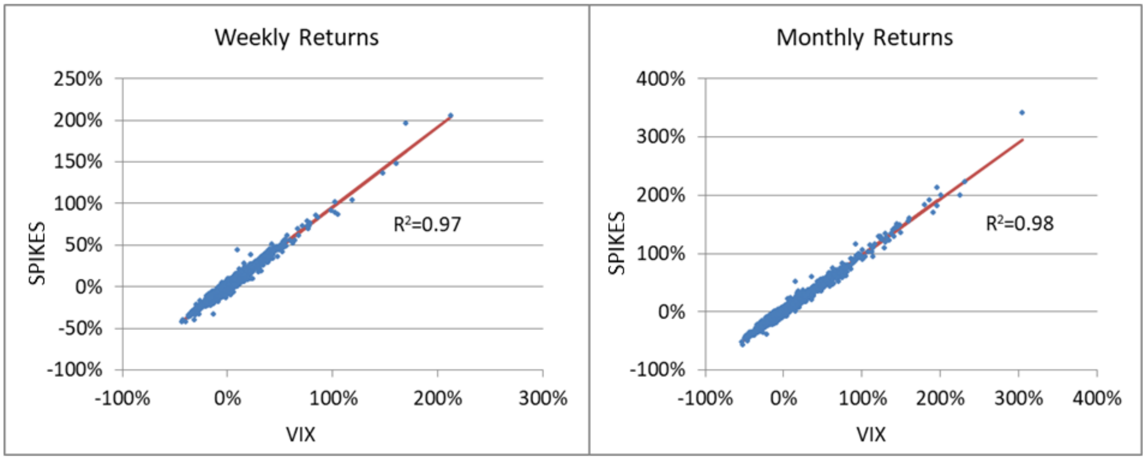

Figure 1: Linear Regressions of SPIKES Returns on VIX Returns

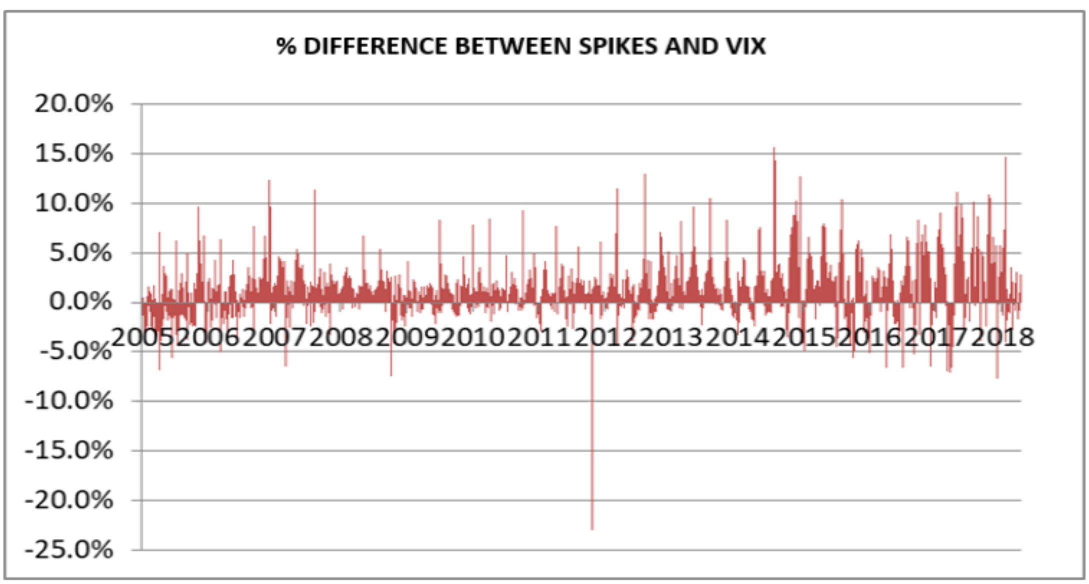

Figure 2: Percentage difference between SPIKES and VIX from 2005 to 2018

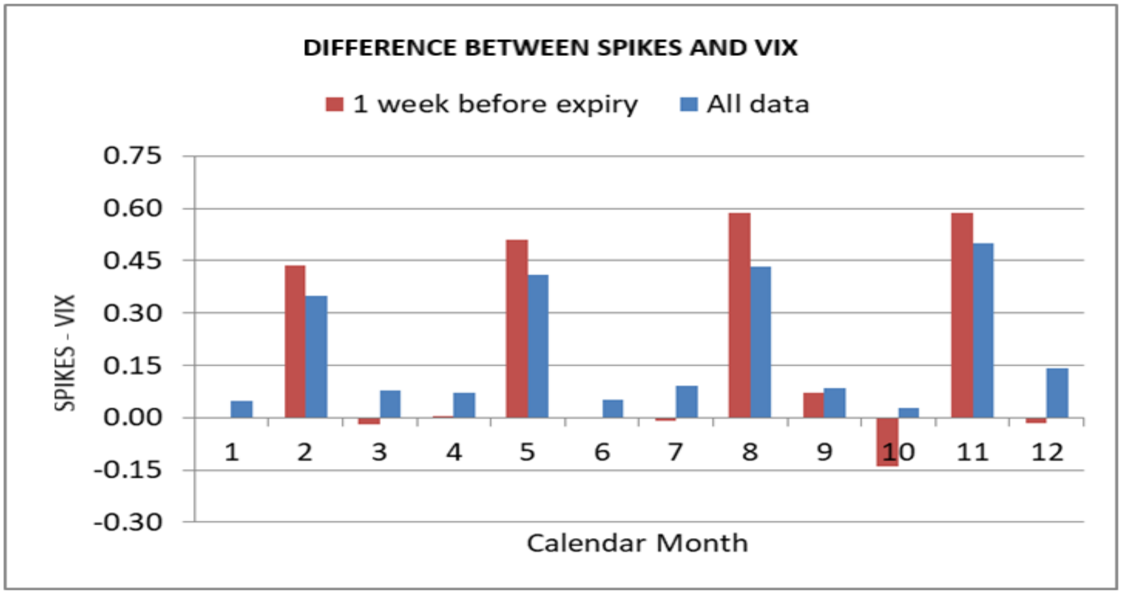

Figure 3: Difference between SPIKES and VIX by calendar months

options combined with dynamic trading in futures written on their underlying index. To ease one's understanding of the construction of the two volatility indices, we first introduce 2 concepts called TVIX and TSPIKES which stand for theoretical VIX and theoretical SPIKES respectiveley.

For $TVIX^2$, the cost of the theoretical replicating portfolio is:

$$TVIX^2 = \frac{365}{30}\frac{1}{B}\left[\int_0^F \frac{2}{K_p^2}p(K_p)dK_p + \int_F^\infty \frac{2}{K_c^2}c(K_c)dK_c\right] \tag{1}$$

where 365/30 is an annualization factor based on calendar days, B is the price of a zero coupon bond paying \$1 in 30 days, and F is the 30 day forward price of S&P500, which is approximated by the futures price in practice. In (1), $p(K_p)$ and $c(K_c)$ respectively denote market prices of 30 day European-style OTM puts and calls written on the S&P500 Index struck at $K_p \in [0, F]$ and $K_c > F$ respectively.

Assuming no frictions, deterministic interest rates, and a strictly positive and continuous futures price process, $TVIX^2$ is the cost of replicating a fictitious variance swap paying the quadratic variation of the log futures price at maturity. Academics sometimes wrongly describe this replication strategy as model-free, but what they should be writing is that this replication is not as model-dependent as the standard approach for replicating path-dependent derivatives. Under either stochastic interest rates and/or jumps in price and/or non-negative futures prices, the terminal quadratic variation of the log futures price cannot be theoretically replicated without further assumptions.

The magnitude of the variance swap replication failure increases as we move from theory towards practice. In practice, the actual variance swap has discrete monitoring, most often daily, and often squares discrete time log price relatives of SPX. not its futures price. Since observed option prices are only at discrete strikes in practice, The above $TVIX^2$ integral formula has to be approximated by either fitting an implied volatility smile across strikes or by replacing the integral with a sum arising from truncation and discrete spacing of strikes. When a sum is used, a correction term is needed to capture the fact that F does not fall on a strike. Since observed option maturities are only rarely exactly 30 days, further approximation error is introduced by the necessity of interpolating across two maturities straddling 30 days. When:

- the annualization factor is based on minutes rather than days
- the integral is replaced by a sum with a correction term
- the maturity interpolation is linear

then the approximation of $TVIX^2$ is called VIX^2. As an aside, one can develop both a theory and a target variance swap like payoff such that VIX^2 without the small correction term is the exact replication cost, as opposed to an approximation of the replication cost of a theoretical or exact variance swap.

TSPIKES is based on the same theoretical weighting scheme as TVIX, but where European-style options on the S&P500 Index are necessarily replaced by American-style options on SPY, which is the S&P500 ETF. Hence:

$$TSPIKES^2 = \frac{365}{30}\frac{1}{B}\left[\int_0^F \frac{2}{K_p^2}P(K_p)dK_p + \int_F^\infty \frac{2}{K_c^2}C(K_c)dK_c\right] \tag{2}$$

where $P(K_p)$ and $C(K_c)$ respectively denote market prices of the American-style OTM puts and calls written on the SPY ETF struck at $K_p \in [0, F]$ and $K_c > F$ respectively. Unlike with $TVIX^2$, there is no known theory under which $TSPIKES^2$ is the initial cost of replicating a theoretical variance swap. The quarterly dividends paid by the SPY ETF can be handled without introducing a model, but current knowledge is such that the early exercise feature requires the introduction of a model to remove the EEP. The TSPIKES integral formula can also be approximated in practice by a better annualization, by truncation and discrete spacing of strikes, and by linear interpolation across two maturities straddling 30 days. This approximation of $TSPIKES^2$ is called $SPIKES^2$.

In variance swap replication theory, the market prices of the European options are observed directly. As a result, the only role of the underlying forward price F is to separate OTM put strikes from OTM call strikes. The underlying forward price is not needed to calculate option premia from a model as the option premia are supposed to be directly observed. Likewise, the American option premia in TSPIKES are supposed to be directly observed. When future values of SPIKES are to be compared to future values of VIX, we don't observe future prices of the component options. However, we can use an option pricing model to project these future option prices onto future relevant stochastic state variables such as spot SPX or SPY, and/or an ATM implied volatility of SPX or SPY. We can then use the model to compare future levels of SPIKES to future levels of VIX, conditional on given numerical values of the relevant state variables.

When an American option on SPY is exercised early or at maturity, its payoff involves the spot value of SPY, not its futures price. When option pricing models are used to determine the EEP of a SPY option, it is easier to evolve the single spot price of the underlying rather than to evolve the entire term structure of forward or futures prices. It is also easier to assume that implied volatilities are constant across moneyness and calendar time rather than to assume they vary with moneyness and stochastically over time.

When we assume that the only relevant stochastic state variable is the spot and that it has constant proportional carrying costs and constant instantaneous volatility over time, we are using the BMS model to price options. Consider first the pricing of European-style SPX options in the BMS model. We assume for the rest of the paper that the dividends from the 500 stocks in SPX are continuously paid over time and that the annualized divided yield of SPX is constant at some known level $\gamma \geq 0$. We also suppose for the rest of this paper that the riskfree interest rate is also constant at some known level $r \geq 0$. Consider the calculation of $TVIX$ in the BMS model, when the current value of the underlying SPX is at some known level $X > 0$, the constant proportional carrying cost is $r - \gamma \in \mathbb{R}$, and the volatility of SPX is constant at some known level $\sigma > 0$:

$$TVIX^2 = \frac{365}{30}\frac{1}{B}\left[\int_0^F \frac{2}{K_p^2} p^{bs}(X, \gamma, K_p) dK_p + \int_F^\infty \frac{2}{K_c^2} c^{bs}(X, \gamma, K_c) dK_c\right], \tag{3}$$

where $p^{bs}(X, \gamma, K_p)$ and $c^{bs}(X, \gamma, K_c)$ respectively denote the BMS model value of a European put and call when SPX is at $X > 0$, with constant continuously paid dividend yield $\gamma \geq 0$, and for strikes $K_p \in [0, F]$ and $K_c \geq F$ respectively.

In this BMS model, $TVIX^2$ is the constant instantaneous variance rate σ^2. Note that $TVIX^2$ is independent of the inputs X, γ, r, and T that enter into the relative pricing of each constituent SPX option. However, when we move from $TVIX^2$ to VIX^2, VIX^2 gains dependence on X, γ, r, and T due to the discreteness of strikes.

Before we move to the pricing of American-style SPY options, consider the theoretical relationship between SPX and SPY when SPX has a constant continuously paid annualized dividend yield $\gamma \geq 0$ and SPY has a constant quarterly paid quarterly compounded annualized dividend yield $q \geq 0$. Consider the one year price relatives $\frac{X_1}{X_0}$ and $\frac{Y_1}{Y_0}$ when time 0 is just after SPY has paid a quarterly dividend. Let D_0 be the price at this time of a pure discount bond paying \$1 in 1 year. Under the forward measure $\mathbb{Q}_1$, we have $D_0 E_0^{\mathbb{Q}_1} \frac{X_1}{X_0} = e^{-\gamma}$, while $D_0 E_0^{\mathbb{Q}_1} \frac{Y_1}{Y_0} = \left(\frac{1}{1+q/4}\right)^4$. Equating the two expressions and solving for q:

$$q = 4(e^{\gamma/4} - 1).$$

$$TSPIKES^2 = \frac{365}{30}\frac{1}{B}\left[\int_0^F \frac{2}{K_p^2} P^{bs}(Y, q, K_p) dK_p + \int_F^\infty \frac{2}{K_c^2} C^{bs}(Y, q, K_c) dK_c\right] \tag{4}$$

where $P^{bs}(Y, q, K_p)$ and $C^{bs}(Y, q, K_c)$ respectively denote the BMS model value of an American put and call when SPY is at $Y > 0$, with constant quarterly paid dividend yield $q \geq 0$, and for strikes $K_p \in [0, F]$ and $K_c \geq F$ respectively.

Subtracting (3) from (4), we have:

$$\begin{aligned} & TSPIKES^2 - TVIX^2 \qquad (5) \\ = & \frac{365}{30}\frac{1}{B}\left[\int_0^F \frac{2}{K_p^2}[P^{bs}(Y, q, K_p) - p^{bs}(X, \gamma, K_p)] dK_p + \int_F^\infty \frac{2}{K_c^2}[C^{bs}(Y, q, K_c) - c^{bs}(X, \gamma, K_c)] dK_c\right] \end{aligned}$$

Let $(p/c)^{bs}(Y, q, K)$ be the theoretical value of a European put or call on SPY. Suppose we subtract and add $(p/c)^{bs}(Y, q, K)$ in (5):

$$TSPIKES^2 - TVIX^2 = \epsilon^x(Y, q, K) + \delta^d(X, Y; \gamma, q, K) \tag{6}$$

where the non-negative SPY option early exercise premium $\epsilon^x(Y, q, K) =$

$$\frac{365}{30}\frac{1}{B}\left[\int_0^F \frac{2}{K_p^2}[P^{bs}(Y, q, K_p) - p^{bs}(Y, q, K_p)] dK_p + \int_F^\infty \frac{2}{K_c^2}[C^{bs}(Y, q, K_c) - c^{bs}(Y, q, K_c)] dK_c\right]$$

and the sign-indefinite dividend timing difference $\delta^d(X, Y; \gamma, q, K) =$

$$\frac{365}{30}\frac{1}{B}\left[\int_0^F \frac{2}{K_p^2}[p^{bs}(Y, q, K_p) - p^{bs}(X, \gamma, K_p)] dK_p + \int_F^\infty \frac{2}{K_c^2}[c^{bs}(Y, q, K_c) - c^{bs}(X, \gamma, K_c)] dK_c\right].$$

Since the SPIKES index is based on a 30 day horizon, there can be at most one quarterly dividend payment from the underlying SPY ETF. When there is no quarterly dividend, the call EEP vanishes. When there is one quarterly proportional dividend there is an explicit exact formula for the EEP of an American call. For no dividends or one quarterly dividend, we also develop an explicit approximation for the EEP of an American put.

4 The Importance of the Early Exercise Premium

In this section, we investigate the magnitude of the early exercise premium embedded in SPY options. The time series analysis has already indicated that the difference between the two volatility indices is highest when the OTM options underlying SPIKES have a higher probability of early exercise.

It is well known that when no dividends are expected before maturity, American call options are never exercised early, As a result, during this time period, the difference between the two volatility indices would be entirely due to the early exercise premium of put options. Carr Jarrow Myneni[2] give the following representation for the initial Black Scholes[1] model value of an American put's EEP with expiry T, strike price K, and a non-dividend paying underlying priced at S_0:

$$e_0(K) = rK \int_0^T e^{-rt} N \left(\frac{\ln\left(\frac{B(t,K)}{S_0}\right) - (r - \sigma^2/2)\,t}{\sigma\sqrt{t}} \right) dt \tag{7}$$

Here, $B(t,K)$ is the deterministic early exercise boundary for strike K and maturity T, which has no known exact formula, but solves an integral equation.

Integrating over all of the OTM strikes $K \in [0, F_0]$ leads to the following difference between SPIKES^2 and VIX^2:

$$TotE_0 = \int_0^{F_0} e_0(K)dK = 2r \int_0^{F_0} \int_0^T \frac{1}{K^2} e^{-rt} N \left(\frac{\log\frac{B(t,K)}{S_0} - (r - \sigma^2/2)\,t}{\sigma\sqrt{t}} \right) dt \tag{8}$$

Theoretically, increasing the level of the short interest rate r raises each American put's EEP, after accounting for the increase in the early exercise boundary $B(t,K), t \in [0,T]$. As a result, the overall effect of an increase in the interest rate r on the cumulative EEP should be positive. To determine the magnitude of this impact, note that the double integral in (8) can only be computed once we have some expression for the early exercise boundary B_t . In the theoretical part of this paper, we will give an analytic expression for the double integral in (8) when approximating each early exercise boundary $B(t,K), t \in [0,T]$ with an exponential function (see Ju [6]).

In Figure 4, T-bill rates from 2003 are plotted.

In general, the EEP embedded in American call and put prices depends not only on the interest rate but also on the dividends, which for SPY are paid quarterly. When the underlying stock pays dividends, American Call options have positive probability of being exercised early. Since SPY pays dividends quarterly, we have to take dividends into account when computing the total difference between SPIKES^2 and VIX^2 .

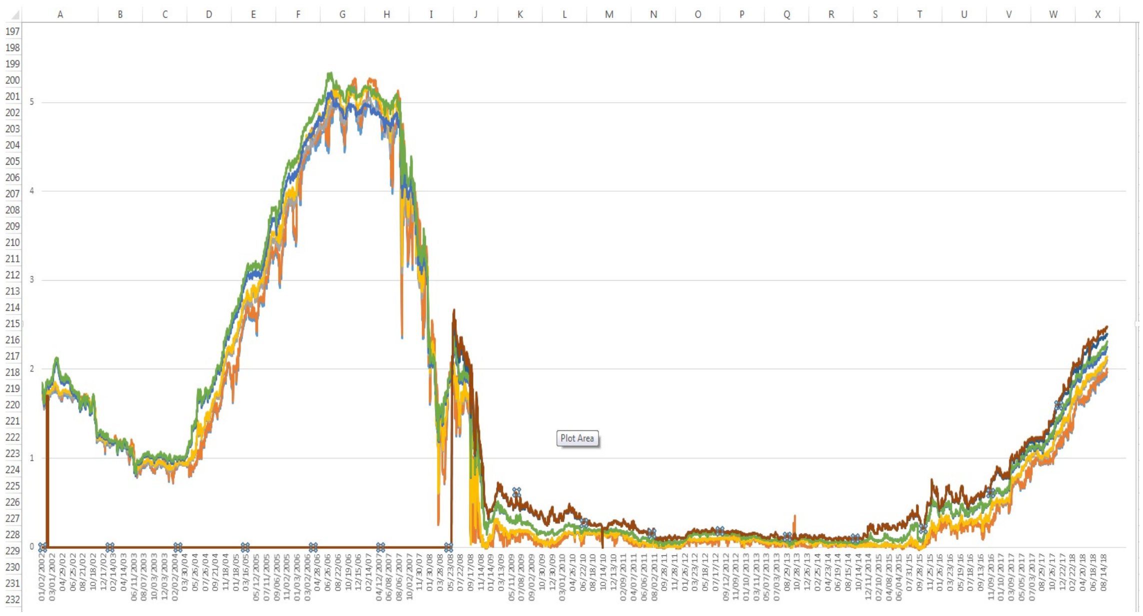

Figure 4: Interest rates (treasury) from 2003 to 2018

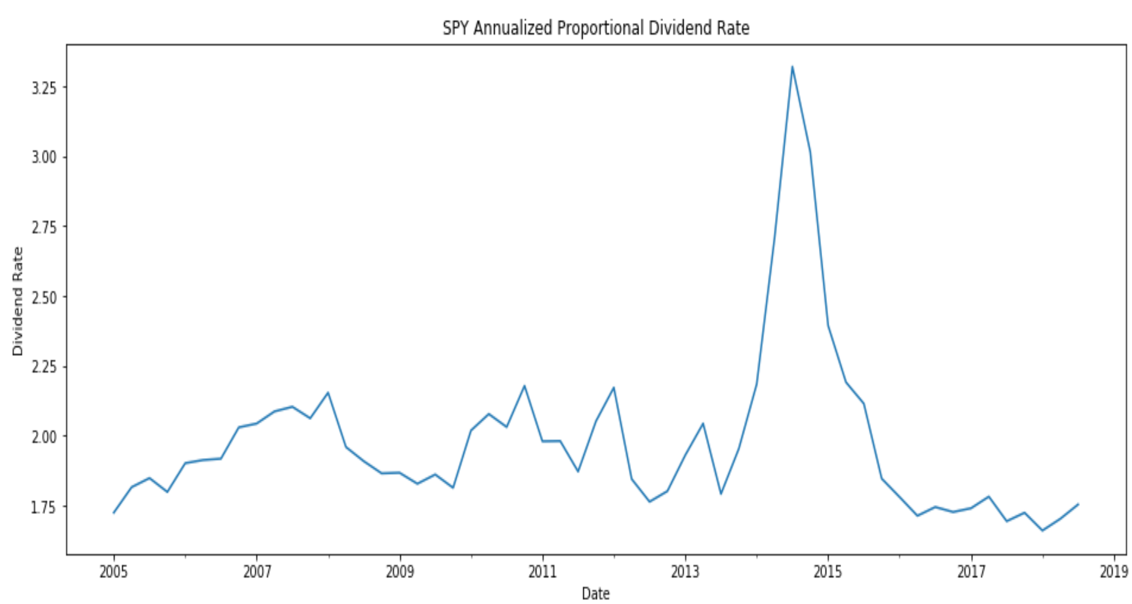

Figure 5: SPY quarterly dividends from March, 31, 2005 to September 28, 2018

In Figure 5, the dividend levels and yields are plotted for SPY since its inception.

We also derive approximate analytic formulas for the difference between SPIKES^2 and VIX^2 under the Black Scholes model, allowing for a discrete proportional dividend paid before maturity. Again, exponential functions are used to approximate the true early exercise boundary (see Ju [6]). To gauge the magnitude of the approximation error, we use finite differences on the Black Scholes PDE with a discrete proportional dividend, qS at T_d, see Gottsche and Vellekop [4], to numerically determine both the early exercise boundary and American option prices. Note that the total difference that we compute will be an upper bound to the actual difference between SPIKES^2 and VIX^2 since the two volatility indices are computed by both truncating the integrals and discretizing strike prices.

5 American Option Pricing When its Underlying ETF Pays a Single Proportional Dividend Before Maturity

In this section, we consider the Black Scholes model when the underlying risky asset pays a single proportional dividend before maturity. The continuously compounded interest rate is constant at $r > 0$. Let S_t be the spot price of this underlying risky asset at time $t \in [0, T]$. Thee risk-neutral dynamics of S are given by:

$$dS_t = rS_t dt + \sigma S_t dWt - \delta(t - T_D)D_t dt$$

where $D_t = qS_t$, q is the proportional dividend rate and δ is the Dirac delta function. Consider American call and put options with expiration date T and strike prices K_c, K_p respectively. It is well known that if the underlying stock pays no dividends between the valuation time t and the options maturity date T, then an American call has the same price as a European one i.e. $C^A(t, T, K_c) = C^E(t, T, K_c)$, where C^E and C^A denote the price of an American and a European call respectively; In contrast, American puts have a positive early exercise premium. We can write

$$P^A(t, T, K_p) = P^E(t, T, K_p) + e_P(t, K_p) \tag{9}$$

where P^E and P^A denote the price of an American and a European put respectively. Here, $e_P(t)$ denotes the early exercise premium at time t of the American put. The put should be exercised at time t if and only if its underlying stock's price is in the exercise region, i.e. if $S_t < S_p(t)$ where $S_p(t)$ denotes the early exercise boundary of the put.

The value $u(t, S_t; K, T)$ of an American option can be shown to solve the following partial differential equation (PDE)

$$\frac{\partial u(t, S_t; K, T)}{\partial t} + (r - \delta(t - T_D)q)S_t \frac{\partial u(t, S_t; K, T)}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 u(t, S_t; K, T)}{\partial S_t^2} = ru(t, S_t; K, T)$$

where T_D is the dividend date. To obtain the correct unique value, boundary conditions must also be applied. To solve this PDE numerically, we used the Crank-Nicholson finite difference scheme.

Note that the above PDE is exactly the same as the PDE for the non-dividend case where we need only set $q = 0$. For the American call option, it is not necessary to numerically solve the above PDE. It can be proven, see [7], that the American call should only be exercised early, if it is sufficiently in the-money just before the dividend is paid (i.e. on the cum-dividend date). Hence, the American call has the same value as a Bermudan call with exercise dates $\{T_d, T\}$. A closed form formula for the American call value is derived in [7] under this setting and is given by:

$$\begin{aligned} C_1^A(t,S,K,T) = (1-q)SN_2 &\left(d_1\left(t,T,(1-q)S,K\right), -d_1\left(t,T_d,S,\overline{S}\right); -\sqrt{\frac{T_d}{T}}\right) \\ -Ke^{-rT}N_2 &\left(d_2\left(t,T,(1-q)S,K\right), -d_2\left(t,T_d,S,\overline{S}\right); -\sqrt{\frac{T_d}{T}}\right) \\ &+SN\left(d_1\left(t,T_d,S,\overline{S}\right)\right) - Ke^{-rT_d}N\left(d_2\left(t,T_d,S,\overline{S}\right)\right), \end{aligned}$$

where $d_1\left(t,T,x,y\right) = \frac{\ln\frac{x}{y}+(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$ and $d_2\left(t,T,x,y\right) = \frac{\ln\frac{x}{y}+(r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$ and $\overline{S}$ is such that

$$\overline{S} - K = C_0^E(t,(1-q)\overline{S},K,T), \tag{10}$$

where $C_0^E(t,S,K,T)$ is the price at time t of an European Style call with no dividends with strike price K, expiration date T on the underlying priced at S. Note that (10) must be solved numerically for $\overline{S}$, using bisection for example. The pricing formula for the American call is in closed form once $\overline{S}$ is obtained. The early exercise premium of a single American call struck at K is given by:

$$e_c(t,K) = e^{-r(T_d-t)}\mathbb{E}_t^Q\left[\left(S_{T_d}-K\right)\mathbf{1}_{\{S_{T_d}>K\}}\right] - e^{-r(T-t)}\mathbb{E}_t^Q\left[\left(S_T-K\right)\mathbf{1}_{\{S_{T_d}>K,S_T>K\}}\right] \tag{11}$$

Straightforward computations give:

$$\begin{aligned} e_c(0,K) &= SN\left(d_1\left(t,T_d,S,\overline{S}\right)\right) - Ke^{-rT_d}N\left(d_2\left(t,T_d,S,\overline{S}\right)\right) \\ -(1-q)SN_2 &\left(d_1\left(t,T,(1-q)S,K\right), d_1\left(t,T_d,S,\overline{S}\right); -\sqrt{\frac{T_d}{T}}\right) \\ +Ke^{-rT}N_2 &\left(d_2\left(t,T,(1-q)S,K\right), d_2\left(t,T_d,S,\overline{S}\right); -\sqrt{\frac{T_d}{T}}\right) \end{aligned}$$

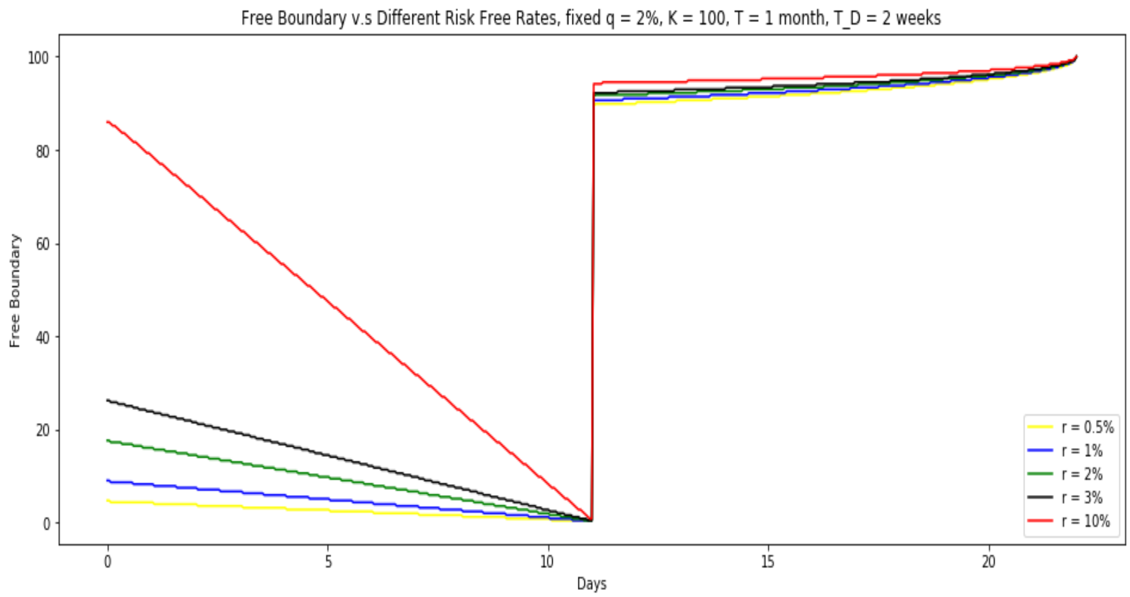

Figure 6: Early exercise boundary curves for different levels of the risk free rate r and fixed dividend rate q.

In the case of an American put, we numerically solve the Black Scholes PDE and obtain the corresponding early exercise boundary. The put prices and exercise boundary obtained by applying the finite difference scheme will be referred as *true values*, even though small errors arise via truncation and discretization.

In Figures 6 and 7, we plot the early exercise boundary for an American put with $T = 1$ month, and $T_d = 1/2$ month, for various values of q and r.

6 Approximating American Put Prices and Exercise Boundary

In this section, we first consider a single American put written on an ETF paying a single discrete proportional dividend qS at the ex-dividend date $T_d < T$. This ex-dividend date is crucial in the definition of the early exercise boundary, which jumps at time T_d. Indeed, after time T_d the option reduces to an American put written on a non-dividend paying asset.

As a first approximation for the American put's Black Scholes value, we compute the price of the corresponding Bermudan put with exercise dates $\{T_d, T\}$. By following a similar proof to that of [7], we can prove that the Black Scholes model value at time t for this Bermudan

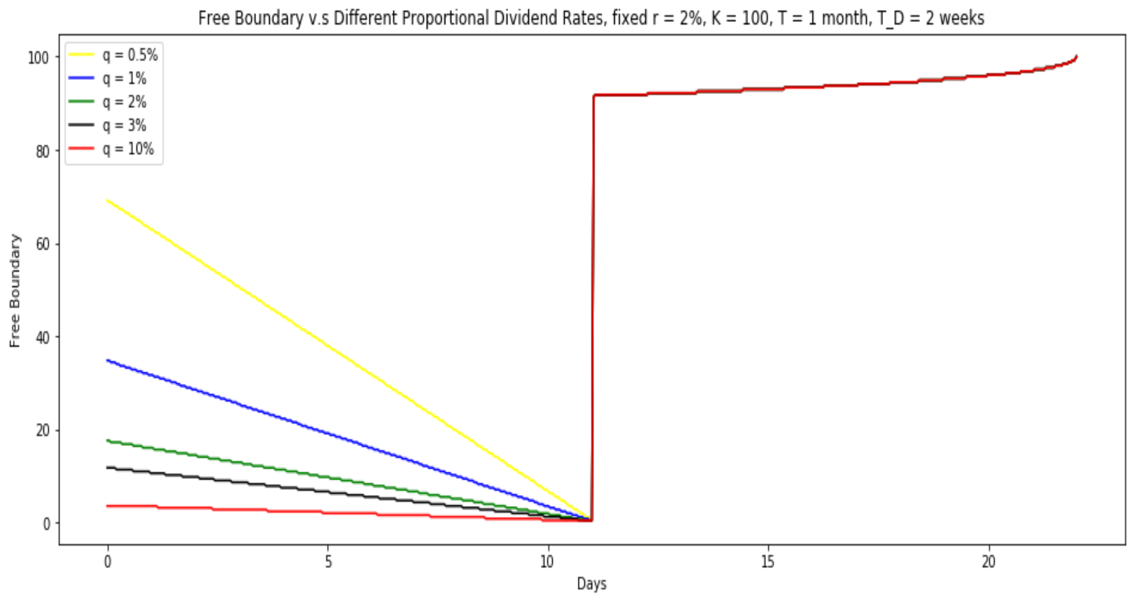

Figure 7: Early exercise boundary curves for different levels of the dividend rate q and fixed risk free rate r.

put is given by:

$$\begin{aligned}P_1^B(t,S,K,T;\underline{S}) = Ke^{-rT}N_2\left(-d_2\left(t,T,(1-q)S,K\right), d_2\left(t,T_d,(1-q)S,\underline{S}\right); -\sqrt{\frac{T_d}{T}}\right)\\ -(1-q)SN_2\left(-d_1\left(t,T,(1-q)S,K\right), d_1\left(t,T_d,(1-q)S,\underline{S}\right); -\sqrt{\frac{T_d}{T}}\right)\\ +Ke^{-rT_d}N\left(d_2\left(t,T_d,(1-q)S,\underline{S}\right)\right) - SN\left(d_1\left(t,T_d,(1-q)S,\underline{S}\right)\right),\end{aligned}$$

where $\underline{S}$ is defined implicitly by

$$K - \underline{S} = P_0^E(t,\underline{S},K,T). \tag{12}$$

Here, $P_0^E(t,S,K,T)$ is the Black Scholes model value at time t of a European put with strike price K, expiration date T, and written on a non-dividend paying underlying asset priced at S. Note that the Bermudan put should only be exercised in T_d, if it is sufficiently in the money, just *after* the dividend is paid (i.e. on the ex-dividend date). Here, $\underline{S}$ is the critical level at T_d for the ex-dividend stock price.

The future value at T of the payoff of the above Bermudan put is given by

$$P^B(T) = e^{r(T-T_d)}\left(K - S_{T_d}\right)\mathbf{1}_{\{S^x_{T_d}\leq\underline{S}\}} + \left(K - S_T\right)^+\mathbf{1}_{\{S^x_{T_d}>\underline{S}\}}, \tag{13}$$

where the superscript x in the stock price at T_d indicates that it is ex-dividend.

Once the critical level $\underline{S}$ is computed by solving (12), then the Bermudan put pricing formula is obtained by taking expectations with respect to the risk-neutral measure Q and discounting the time T payoff in (13).

When the Bermudan put pricing formula is used to approximate the American put price, we are implicitly assuming that the early exercise boundary is flat at 0 after the dividend is paid. Instead, the true boundary increases as t approaches T towards the strike price K; The early exercise boundary is that of an American put written on a non-dividend paying stock in $[T_d, T]$ We will assume in what follows that the early exercise boundary is approximated by an exponential function of time, both before and after the ex dividend date. Under this assumption, we derive a closed-form formula for the early exercise premium of an American put. Hence, the true early exercise boundary $S_p = \{S^p(t), 0 < t < T\}$ is approximated by

$$S_p(t) \approx \begin{cases} Ce^{ht}, & \text{if} \quad 0 < t < T_d \\ \\ Le^{gt}, & \text{if} \quad T_d < t < T \end{cases} \tag{14}$$

where C, L, h, g are constant parameters. The approximation for the boundary is decreasing in calendar time t before time T_d, and increasing in t afterwards. The presence of a discrete dividend makes our approximation of the boundary jump upward at time T_d . The above exponential specification induces an approximate linear behaviour before time T_d if h has a small absolute value or if $T_d << 1$ as is the case here where $T_d < T \approx 1/12$.

We proceed in three steps:

1. we first restrict attention to the case $t > T_d$ and consider a constant boundary L by setting $g = 0$.
2. we then apply a change of probability measure to derive a closed form pricing formula in the case of an exponential growing boundary
3. we repeat the 2 steps above for the case when $t < T_d$.

6.0.1 Step 1: Constant Boundary after Time T_d

Since optimal early exercise is possible in the time interval $[T_d, T]$, the American put value in this time interval is the sum of the European Put price and the early exercise premium. Since there are no dividends in the time interval $[T_d, T]$, this early exercise premium is just the present value of the interest earned on the strike price while the stock price is below the early exercise boundary in the time interval $[T_d, T]$, as shown in [2] .

Hence, the price at time $t < T_d$ of this hybrid American put is given by:

$$P_1^H(t, S, K, T; L) = P_1^E(t, S_t, K, T, L) + e^{-r(T_d - t)}\mathbb{E}^Q\left[rK\int_{T_d}^{T} e^{-r(u-T_d)}\mathbf{1}_{\{S_u < S_p(u)\}}du\right]. \quad (15)$$

The critical stock price at time T_d is defined as the level for S such that the continuation value of the hybrid put equals its exercise value; if we assume that the exercise boundary is constant at this critical stock price after time T_d, then L implicitly solves:

$$P_1^H(T_d, L, K, T; L) = K - L. \quad (16)$$

We know that when the underlying asset is non-dividend paying, the value at time $t \in [T_d, T]$, of interest on the strike while $S < L$ is

$$\mathbb{E}_t^Q\left[rK\int_t^T e^{-r(u-t)}\mathbf{1}_{\{S_u < L\}}du\right]. \quad (17)$$

Consider a function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $f \in \mathcal{C}^2$. By applying Ito's Lemma in integral form we get

$$\begin{aligned}
e^{-rT}f(S_T) = f(S_t) + \int_t^T e^{-r(u-t)}f'(S_u)dS_u + \int_t^T e^{-r(u-t)}[\frac{f''(S_u)}{2}\sigma^2 S_u^2 - rf(S_u)]du \\
= f(S_t) + \int_t^T e^{-r(u-t)}f'(S_u)[dS_u - rS_u du] \\
+ \int_t^T e^{-r(u-t)}[\frac{f''(S_u)}{2}\sigma^2 S_u^2 + rS_u f'(S_u) - rf(S_u)]du
\end{aligned}$$

and, taking conditional expectation at time $t \in [T_d, T]$,

$$e^{-r(T-t)}\mathbb{E}_t^Q\left[f(S_T)\right] = f(S_t) + \mathbb{E}_t^Q\left[\int_t^T e^{-r(u-t)}[\frac{f''(S_u)}{2}\sigma^2 S_u^2 + rS_u f'(S_u) - rf(S_u)]du\right]. \quad (18)$$

The value of the accrued interest on the strike for $S < L$ may be written, for $t \in [T_d, T]$ as

$$\mathbb{E}_t^Q\left[rK\int_t^T e^{-r(u-t)}\mathbf{1}_{\{S_u < L\}}du\right] = f(S_t) - e^{-r(T-t)}\mathbb{E}_t^Q\left[f(S_T)\right], \quad (19)$$

if and only if there exists a function $f(S)$ solving the following ordinary differential equation (ODE):

$$\frac{f''(S_u)}{2}\sigma^2 S_u^2 + rS_u f'(S_u) - rf(S_u) = -rK\mathbf{1}_{\{S_u < L\}}. \quad (20)$$

It is straightforward to show that a solution exists for any constant L, and it is given by

$$f(S;L) = \frac{K}{r+\frac{\sigma^2}{2}}\left[\mathbf{1}_{\{S<L\}}\left(r+\frac{\sigma^2}{2}-r\frac{S}{L}\right)+\mathbf{1}_{\{S>L\}}\frac{\sigma^2}{2}\left(\frac{S}{L}\right)^{-2r/\sigma^2}\right]. \tag{21}$$

The function $f(S;L)$ can be seen as a final payoff at T whose time value at t matches the value of the interest earned on the strike price K when $S<L$ between t and T. Hence, the price of the Hybrid Put with constant boundary at time t, for $0 \le t \le T_d$, is

$$P_1^H(t,S_t,K,T;L) = P_1^E(t,S_t,K,T) + e^{-r(T_d-t)}\mathbb{E}_t^Q\left[f(S_{T_d}) - e^{-r(T-T_d)}f(S_T)\right]. \tag{22}$$

To solve for L, we must compute the above price at time T_d i.e. conditioning on the information available at time T_d and then solving (16). Note that the value for L depends on the strike price K of the option i.e. $L = bK$ for some positive constant b.

The early exercise premium at time T_d is given by:

$$\begin{aligned}
e_p(T_d,K) = \mathbb{E}_{T_d}^Q&\left[f(S_{T_d}) - e^{-r(T-T_d)}f(S_T)\right] = \\
&\frac{K}{r+\frac{\sigma^2}{2}}\left[(r+\frac{\sigma^2}{2})\left[\mathbf{1}_{\{S_{T_d}<L\}} - e^{-r(T-T_d)}N\left(-d_2\left(T_d,S_{T_d},L,T\right)\right)\right]\right. \\
&\left.-\frac{rS_{T_d}}{L}\left[\mathbf{1}_{\{S_{T_d}<L\}} - N\left(-d_1\left(T_d,S_{T_d},L,T\right)\right)\right]\right] \\
&+\frac{K}{r+\frac{\sigma^2}{2}}\frac{\sigma^2}{2}\left(\frac{S_{T_d}}{L}\right)^{\frac{-2r}{\sigma^2}}\left[\mathbf{1}_{\{S_{T_d}>L\}} - N\left(d_2\left(T_d,S_{T_d},L,T\right)-\frac{2r}{\sigma}\sqrt{T-T_d}\right)\right]
\end{aligned}$$

and the early exercise premium $e_p(0,K)$ at time $t=0$ is the discounted expected value of $e_p(T_d)$. It can be computed in a similar fashion and is given by

$$\begin{aligned}
e_p(0,K) = {}& K\left[e^{-rT_d}N\left(-d_2\left(0,S_0(1-q),L,T_d\right)\right) - e^{-r(T-T_d)}N\left(-d_2\left(0,S_0(1-q),L,T\right)\right)\right] \\
&-\frac{K}{r+\frac{\sigma^2}{2}}\frac{rS_0(1-q)}{L}\left[N\left(-d_1\left(0,S_0(1-q),L,T_d\right)\right) - N\left(-d_1\left(0,S_0(1-q),L,T\right)\right)\right] \\
&+\frac{K}{r+\frac{\sigma^2}{2}}\frac{\sigma^2}{2}\left(\frac{S_0(1-q)}{L}\right)^{\frac{-2r}{\sigma^2}}\left[N\left(d_2\left(0,S_0(1-q),L,T_d\right)-\frac{2r}{\sigma}\sqrt{T_d}\right)\right. \\
&\left.-N\left(d_2\left(0,S_0(1-q),L,T\right)-\frac{2r}{\sigma}\sqrt{T}\right)\right].
\end{aligned}$$

Finally, the price at time $t=0$ for the hybrid put is given by:

$$P_1^H(0,S_0,K,T;L) = P_1^E(0,S_0,K,T) + e_p(0,K). \tag{23}$$

6.0.2 Step 2: Exponential Boundary after time T_d

As an improved approximation, we can let the exponential growth coefficient g be non-zero; in this case it is possible to obtain a closed form formula for the early exercise premium according to the outcomes in [6]. In this paper, the constants L, g are computed by imposing the *value matching* and *smooth-pasting* conditions. Here, we just impose value matching at time T_d. We set the terminal boundary level at K when $t = T$. Define $\tilde{S}_t := S_t e^{-gt}$. Under the risk-neutral probability measure Q, the dynamics of $\tilde{S}$ are given by:

$$d\tilde{S}_t = (r-g)\tilde{S}_t dt + \sigma \tilde{S}_t dWt - \delta(t - T_D)D_t dt.$$

We can use a similar approach to that of a constant boundary and find a function $\tilde{f}$ such that

$$\mathbb{E}_t^Q\left[rK\int_t^T e^{-r(u-t)}\mathbf{1}_{\{\tilde{S}_u<L\}}du\right] = \tilde{f}(\tilde{S}_t) - e^{-r(T-t)}\mathbb{E}_t^Q\left[\tilde{f}(\tilde{S}_T)\right], \tag{24}$$

which is obtained by solving:

$$\frac{\tilde{f}''(\tilde{S}_u)}{2}\sigma^2\tilde{S}_u^2 + (r-g)\tilde{S}_u\tilde{f}'(\tilde{S}_u) - r\tilde{f}(\tilde{S}_u) = -rK\mathbf{1}_{\{S_u<L\}}. \tag{25}$$

We get:

$$\tilde{f}(\tilde{S};L,g) = \frac{2rK}{\sigma^2(p_+-p_-)}\left[\mathbf{1}_{\{\tilde{S}<L\}}\left(\frac{1}{p_+}-\frac{1}{p_-}-\frac{1}{p_+}\left(\frac{\tilde{S}}{L}\right)^{p_+}\right) - \mathbf{1}_{\{\tilde{S}>L\}}\frac{1}{p_-}\left(\frac{\tilde{S}}{L}\right)^{p_-}\right], \tag{26}$$

where p_+, p_- are the roots of $\mathcal{P}(p) = \frac{\sigma^2}{2}p^2 + (r-g-\frac{\sigma^2}{2})p - r$.

Finally:

$$\begin{aligned}
\tilde{e}_p(T_d,K) = \mathbb{E}_{T_d}^Q&\left[\tilde{f}(\tilde{S}_{T_d}) - e^{-r(T-T_d)}\tilde{f}(\tilde{S}_T)\right] = \\
&\frac{2rK}{\sigma^2(p_+-p_-)}\left[(\frac{1}{p_+}-\frac{1}{p_-})\left[\mathbf{1}_{\{\tilde{S}_{T_d}<L\}} - e^{-r(T-T_d)}N\left(-d_2\left(T_d,\tilde{S}_{T_d},L,T\right)\right)\right]\right. \\
&\left.-\frac{1}{p_+}\left(\frac{\tilde{S}_{T_d}}{L}\right)^{p_+}\left[\mathbf{1}_{\{\tilde{S}_{T_d}<L\}} - N\left(-d_2\left(T_d,\tilde{S}_{T_d},L,T\right) - p_+\sigma\sqrt{T-T_d}\right)\right]\right] \\
&-\frac{2rK}{\sigma^2(p_+-p_-)}\frac{1}{p_-}\left(\frac{\tilde{S}_{T_d}}{L}\right)^{p_-}\left[\mathbf{1}_{\{\tilde{S}_{T_d}>L\}} - N\left(d_2\left(T_d,\tilde{S}_{T_d},L,T\right) + p_-\sigma\sqrt{T-T_d}\right)\right],
\end{aligned}$$

and similarly we obtain the early premium at time $t = 0$.

In Figure 8, we plot the early exercise boundary suggested by finite differences along with our two suggested approximations for $r = 2\%, q = 3\%$ and $T_d = 1/2$ month.

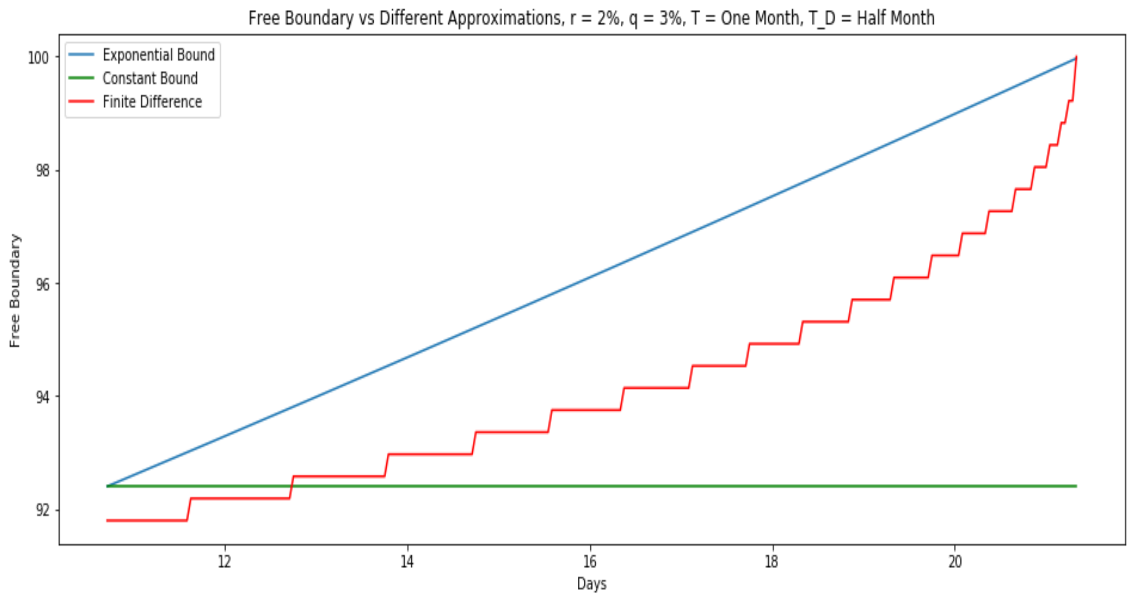

Figure 8: Early exercise boundary after the dividend date T_d: Finite difference (Red), Constant approximation (Green), Exponential approximation (Blue)

6.0.3 Step 3: Exponential Boundary Approximation in $[0, T]$

Consider now that early exercise is also possible within the time interval $[0, T_d]$ just before the dividend is paid and assume that the boundary is approximated as in (14). The discrete characteristics of the dividend make it possible to write the early exercise premium as:

$$\begin{aligned} & \mathbb{E}_0^Q \left[rK \int_0^{T_d} e^{-ru} \mathbf{1}_{\{S_u^c < Ce^{hu}\}} du + rK \int_{T_d}^{T} e^{-ru} \mathbf{1}_{\{S_u^x < Le^{gu}\}} du \right] \\ = \; & \mathbb{E}_0^Q \left[rK \int_0^{T_d} e^{-ru} \mathbf{1}_{\{S_u^c < Ce^{hu}\}} du \right] + \mathbb{E}_0^Q \left[rKe^{-rT_d} \int_{T_d}^{T} e^{-r(u-T_d)} \mathbf{1}_{\{S_u^x < Le^{gu}\}} du \right]. \quad (27) \end{aligned}$$

Note that the dividend only enters the above formula because the cum-dividend stock price appears in the indicator function in the first integral, while the ex-dividend stock price appears in the indicator function in the second integral. Hence, similar computations can be derived when the boundary is approximated before T_d. Once the values for C, h are computed, then the overall early exercise premium for an American put with strike K and expiration date T, can be computed as

$$\begin{aligned} & e_p(0, K) = \\ & \mathbb{E}_0^Q \left[f(S_0^c; C, h) - e^{-rT_d} f(S_{T_d}^c e^{-hT_d}; C, h) + e^{-rT_d} f(S_{T_d}^x e^{-gT_d}; L, g) - e^{-rT} f(S_T^x e^{-gT}; L, g) \right]. \end{aligned}$$

The expected values are calculated as in Step 2 using the cum-dividend initial price for the first two terms. From Figures 6 and 7, it is evident that the behavior of the boundary is

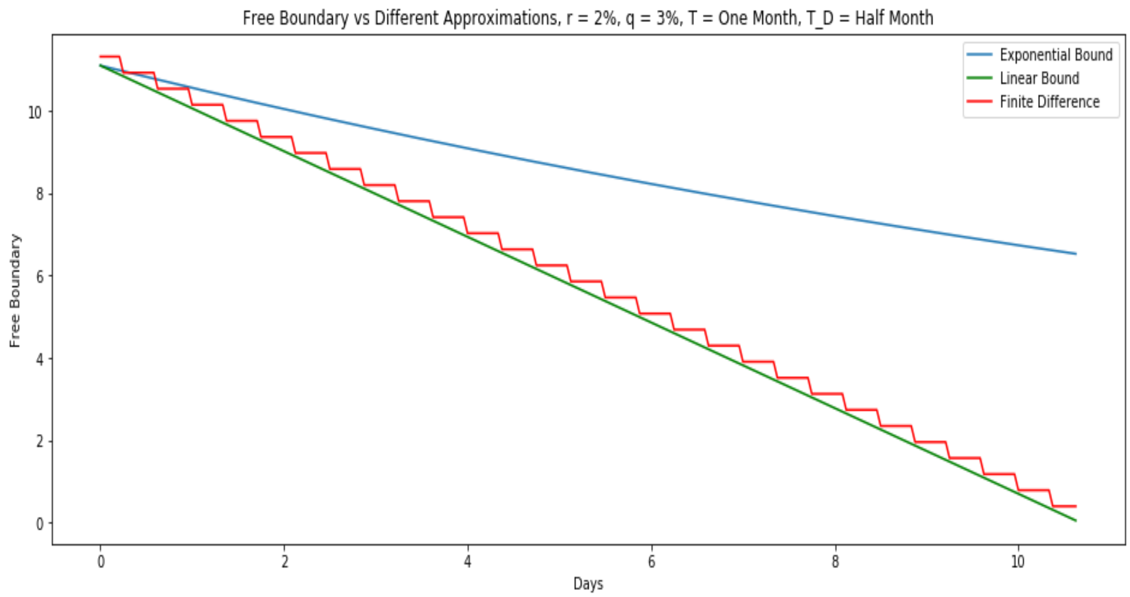

Figure 9: Early exercise boundary before the dividend date T_d: Finite difference (Red), Linear approximation (Green), Exponential approximation (Blue)

essentially linear before time T_d and vanishes at time T_d. By looking at the numerical results, we surmised that the intercept is approximately at the level $\frac{r}{q}KT_d$. A linear boundary approximation with the above properties would be $y = \frac{r}{q}K(T_d - t)$. An exponential function approximating this linear function is obtained by setting $C = \frac{r}{q}KT_d$ and $h = \frac{1}{T_d}$. Another possible choice is available by matching the average slope of the area under the linear and exponential boundary approximations.

In Figure 9, we plot the early exercise boundary from finite differences, along with our suggested linear function and its corresponding exponential approximation.

7 The Approximate Cumulative EEP for American Puts

Recall the representation (21) of the EEP of a single American put as the value of the interest earned on the strike price while the stock price is in the stopping region. This representation allows us to compute directly the excess of $SPIKES^2$ over VIX^2 arising from the EEP of all of the OTM puts. Suppose for simplicity that we consider the early exercise premium arising only from the time interval $[T_d, T]$. Define $\tilde{G}(\tilde{S}; b, g) = \int_0^{F_0} f(\tilde{S}; bK, g)dK$ where the boundary L has been expressed as proportional to the strike price of the option. Then integrating the early premium over the the portfolio of OTM puts, we obtain the total contribution of the

OTM puts to $SPIKES^2$:

$$TotE_p(0) = \int_0^{F_0} \frac{2}{K^2} e_p(0,K)dK = \mathbb{E}^Q[e^{-rT_d} G(\tilde{S}_{T_d}; b, g) - e^{-rT} G(\tilde{S}_T; b, g)]. \tag{28}$$

By integrating the function $\tilde{f}(\tilde{S}; bK, g)$ with respect to K, we obtain:

$$\tilde{G}(\tilde{S}; b, g) = \frac{4r}{\sigma^2(p^+ - p^-)} \left[\mathbf{1}_{\{\tilde{S} < bF_0\}} \left(\left(\frac{1}{p^+} - \frac{1}{p^-} \right) \log \frac{F_0 b}{\tilde{S}} + \frac{1}{(p^+)^2} \left(\frac{\tilde{S}^{p^+}}{(F_0 b)^{p^+}} - 1 \right) + \frac{1}{(p^-)^2} \right) \right.$$
$$\left. + \mathbf{1}_{\{\tilde{S} > bF_0\}} \frac{\tilde{S}^{p^-}}{(bF_0)^{p^-}} \right].$$

After discounting and taking expectations, we get $\mathbb{E}^Q \left[e^{-rt} \tilde{G}(\tilde{S}_t) \right] =$

$$2e^{-rt} \left[\left(\log \frac{F_0 e^{gt} b}{S_0(1-q)e^{r-\sigma^2/2}} - \frac{2r}{\sigma^2(p^+ - p^-)} \left(\frac{1}{(p^+)^2} - \frac{1}{(p^-)^2} \right) \right) N\left(-d_2(0, t, S_0(1-q), F_0 b e^{gt})\right) \right.$$
$$\left. + \sigma\sqrt{t} N'\left(-d_2(0, t, S_0(1-q), F_0 b e^{gt})\right) \right]$$
$$+ \frac{4r}{\sigma^2(p^+ - p^-)} \frac{1}{(p^+)^2} \left(\frac{S_0(1-q)}{F_0 b} \right)^{p^+} N\left(-d_2(0, t, S_0(1-q), F_0 b e^{gt}) - \sigma p^+ \sqrt{t} \right)$$
$$+ \frac{4r}{\sigma^2(p^+ - p^-)} \frac{1}{(p^-)^2} \left(\frac{S_0(1-q)}{F_0 b} \right)^{p^-} N\left(d_2(0, t, S_0(1-q), F_0 b e^{gt}) + \sigma p^- \sqrt{t} \right).$$

The total contribution of the EEP from the American calls should also be added in order to obtain the overall approximated difference between $SPIKES^2$ and VIX^2. This can be done easily since the EEP for American calls is just the difference in value between a Bermudan call and a European call, which is available in closed form.

8 Numerical Results

In this section, we compute the difference between $SPIKES^2$ and VIX^2, and the difference between $SPIKES - VIX$ according to the values obtained via a Crank Nicolson finite difference scheme. We consider four different cases, the dividend is paid one week from now, two weeks from now, three weeks from now, and no dividend is paid by maturity, which is one month in this example. For each case, we fix $S_0 = 100$, $\sigma = 0.2$, and $T =$ one month. We let r and q vary to test how interest rates and dividend yields affect the difference between $SPIKES^2$ and VIX^2, and the difference between $SPIKES$ and VIX.

To calculate the European option prices used in the VIX^2 computation, we linked the annualized continuously paid dividend yield δ to the annualized quarterly proportion q used for $SPIKES^2$ via $\delta = -\log(1-q)$. The integration range for strikes with spot at $S_0 = 100$ is truncated at $K_1 = 30$ and $K_2 = 200$.

In Table 3 and Table 4, we report examples for the value of the total difference between $SPIKES^2$ and VIX^2 and the total difference between $SPIKES$ and VIX for several pairs of r and q. We find that when q is very small, 1%, the total difference for both $SPIKES^2 - VIX^2$ and $SPIKES - VIX$ is negative. When r is fixed, increasing q increases the total premium.

SPIKES-VIX Closed Form	r	q	No. div.	$T_d = 1$ week	$T_d = 2$ weeks	$T_d = 3$ weeks
	0.01	0.01	1.30E-05	-0.000412626	-0.000267007	-7.43E-06
	0.01	0.03	-3.66E-05	0.000605388	0.000895968	0.001314416
	0.01	0.05	-0.000138456	0.001479321	0.002113966	0.002837513
	0.01	0.1	-0.000642321	0.005388662	0.006800814	0.008049942
	0.03	0.01	6.84E-05	-0.000417121	-0.000299388	-5.03E-05
	0.03	0.03	1.87E-05	0.000591923	0.000846527	0.001260788
	0.03	0.05	-8.32E-05	0.001442414	0.002036384	0.002767697
	0.03	0.1	-0.00058665	0.005298913	0.006696017	0.007975492
	0.05	0.01	0.000144085	-0.000414529	-0.000307816	-7.28E-05
	0.05	0.03	9.45E-05	0.000589764	0.000823671	0.001228497
	0.05	0.05	-7.21E-06	0.001423155	0.001988289	0.002720695
	0.05	0.1	-0.00051032	0.005240393	0.006629515	0.007928368
	0.1	0.01	0.000407422	-0.000365409	-0.000226749	-5.29E-05
	0.1	0.03	0.000357488	0.000650452	0.000877054	0.001224064
	0.1	0.05	0.0002564	0.001474059	0.001987673	0.002680582
	0.1	0.1	-0.000245625	0.005264258	0.006607929	0.00789735

Table 3: SPIKES-VIX computed with the approximation formulas

One naturally wonders whether OTM calls or OTM puts are more important in explaining the cumulative EEP across strikes. In the Appendix, we collect all outcomes for the separate contributions of OTM calls and OTMs put to the total EEP and hence to the difference between $SPIKES^2$ and VIX^2.

SPIKES-VIX Finite Difference	r	q	No. div.	$T_d = 1$ week	$T_d = 2$ weeks	$T_d = 3$ weeks
	0.01	0.01	0.000013	-0.000413	-0.000267	-0.000007
	0.01	0.03	-0.000037	0.000605	0.000896	0.001314
	0.01	0.05	-0.000138	0.001479	0.002114	0.002838
	0.01	0.10	-0.000642	0.005389	0.006801	0.008050
	0.03	0.01	0.000068	-0.000417	-0.000299	-0.000050
	0.03	0.03	0.000019	0.000592	0.000847	0.001261
	0.03	0.05	-0.000083	0.001442	0.002036	0.002768
	0.03	0.10	-0.000587	0.005299	0.006696	0.007975
	0.05	0.01	0.000144	-0.000415	-0.000308	-0.000073
	0.05	0.03	0.000095	0.000590	0.000824	0.001228
	0.05	0.05	-0.000007	0.001423	0.001988	0.002721
	0.05	0.10	-0.000510	0.005240	0.006630	0.007928
	0.10	0.01	0.000407	-0.000365	-0.000227	-0.000053
	0.10	0.03	0.000357	0.000650	0.000877	0.001224
	0.10	0.05	0.000256	0.001474	0.001988	0.002681
	0.10	0.10	-0.000246	0.005264	0.006608	0.007897

Table 4: Difference between $SPIKES$ and VIX, according to Finite Difference for $S_0 = 100$, $\sigma = 0.2$ by letting r and q vary and when there is no dividend in the next month or one dividend paid in 1,2 or 3 weeks respectively.

9 Computing the Vega of SPIKES

Again, we proceed in steps.

9.0.1 Vega of an American Put

In this section, we compute the derivative with respect $v = \sigma^2$ of the early exercise premium of an American put with maturity T and strike K, assuming an exponential early exercise boundary $\underline{S}_t = Le^{gt}$. For a put, the calculation applies when no dividends are paid or in the case of a single dividend, for the period just after the dividend date.

If T is large, then piecewise approximation of the early exercise boundary via a sequence of exponential functions is a better approximation. Hence, the approach used here can be easily extended.

Assume that a single dividend is to be paid at T_d. The early exercise premium at time $t = 0$ of an American put with strike K and maturity T, relative to the period $[T_d, T]$, is given by (NOTE THAT IN ORDER TO OBTAIN THE OVERALL PREMIUM WE HAVE TO ADD THE PREMIUM FOR EXERCISING BEFORE T_d).

$$\begin{aligned}
\tilde{e}_p(0, K) &= e^{-rT_d}\mathbb{E}^Q\left[\tilde{f}(\tilde{S}_{T_d}) - \mathbb{E}^Q_{T_d}\left[e^{-r(T-T_d)}\tilde{f}(\tilde{S}_T)\right]\right] \\
&= \mathbb{E}^Q\left[e^{-rT_d}\tilde{f}(\tilde{S}_{T_d})\right] - \mathbb{E}^Q\left[e^{-rT}\tilde{f}(\tilde{S}_T)\right]
\end{aligned}$$

where we have

$$\begin{aligned}
\mathbb{E}^Q\left[e^{-rt}\tilde{f}(\tilde{S}_t)\right] &= Ke^{-rt}N\left(-d_2(0, S_0(1-q), Le^{gt}, t)\right) \\
&- \frac{2rK}{\sigma^2(p_+ - p_-)}\frac{1}{p_+}\left(\frac{S_0(1-q)}{L}\right)^{p_+} N\left(-d_2^+(0, S_0(1-q), Le^{gt}, t)\right) \\
&- \frac{2rK}{\sigma^2(p_+ - p_-)}\frac{1}{p_-}\left(\frac{S_0(1-q)}{L}\right)^{p_-} N\left(d_2^-(0, S_0(1-q), Le^{gt}, t)\right)
\end{aligned}$$

since $\frac{2r}{\sigma^2(p_+ - p_-)}\left(\frac{1}{p_+} - \frac{1}{p_-}\right) = 1$ and where $d_2^-(0, S_0, Le^{gt}, t) = d_2(0, S_0, Le^{gt}, t) + p^-\sigma\sqrt{t}$ and $d_2^+(0, S_0, Le^{gt}, t) = d_2(0, S_0, Le^{gt}, t) + p^+\sigma\sqrt{t}$.

The above value is the sum of three terms that we denote by $I(\sigma, t), II(\sigma, t), III(\sigma, t)$. Computing the partial derivative of each term w.r.t. σ, we get:

$$\frac{\partial}{\partial\sigma}I(\sigma, t) = -Ke^{-rt}n\left(-d_2(0, S_0, Le^{gt}, t)\right)\frac{\partial d_2(0, S_0, Le^{gt}, t)}{\partial\sigma}$$

$$\frac{\partial}{\partial\sigma}II(\sigma,t) = -\frac{\partial}{\partial\sigma}\left(\frac{2rK}{\sigma^2(p_+ - p_-)}\frac{1}{p_+}\left(\frac{S_0}{L}\right)^{p_+}\right) N\left(-d_2^+(0, S_0, Le^{gt}, t)\right)$$
$$+ \frac{2rK}{\sigma^2(p_+ - p_-)}\frac{1}{p_+}\left(\frac{S_0}{L}\right)^{p_+} n\left(-d_2^+(0, S_0, Le^{gt}, t)\right)\frac{\partial d_2^+(0, S_0, Le^{gt}, t)}{\partial\sigma}$$

and:

$$\frac{\partial}{\partial\sigma}III(\sigma,t) = -\frac{\partial}{\partial\sigma}\left(\frac{2rK}{\sigma^2(p_+ - p_-)}\frac{1}{p_-}\left(\frac{S_0}{L}\right)^{p_-}\right) N\left(d_2^-(0, S_0, Le^{gt}, t)\right)$$
$$- \frac{2rK}{\sigma^2(p_+ - p_-)}\frac{1}{p_-}\left(\frac{S_0}{L}\right)^{p_-} n\left(d_2^-(0, S_0, Le^{gt}, t)\right)\frac{\partial d_2^-(0, S_0, Le^{gt}, t)}{\partial\sigma}$$

Summing the above three terms leads to the desired partial derivative, which is hence available in closed form, although the exact computation is quite long and tedious. The derivative with respect to σ of the early exercise premium relative to the life of the option after T_d is obtained as the difference:

$$\frac{\partial}{\partial\sigma}\tilde{e}_p(0,K) = \frac{\partial}{\partial\sigma}I(\sigma,T_d)+\frac{\partial}{\partial\sigma}II(\sigma,T_d)+\frac{\partial}{\partial\sigma}III(\sigma,T_d)-!\frac{\partial}{\partial\sigma}I(\sigma,T)+\frac{\partial}{\partial\sigma}II(\sigma,T)+\frac{\partial}{\partial\sigma}III(\sigma,T) \tag{29}$$

In Tables 9,10,11,12 in the Appendix, we exhibit the differences between SPIKES Vega and VIX Vega for various values of parameters r, q and σ.

10 Conclusions and Future Research

When SPIKES is back-calculated to 2005, it hardly differs from VIX. We used the benchmark BMS model to assess whether this negligible difference would continue. So long as 30 day US interest rates and annualized dividend yields continue to be range bound between 0 and 10 % per year, we conclude that this negligible difference will continue. The prices of the near and next maturity OTM options used to calculate both SPIKES and VIX respond primarily to volatility, not interest rates and dividends. While the EEP embedded in American put and call prices have some sensitivity to these carrying costs, an increase in either rate leads to a mixed response in the EEP of puts and calls.

Future theoretical research can investigate whether these initial conclusions are robust to the assumptions made in the benchmark BMS model that interest rates, dividend yields, and variance rates are constant. When all three of these rates are allowed to be stochastic, the extra optionality in an American option gives its holder the right to swap an exposure based primarily on volatility for a swap between interest and dividends. The EEP should rise due to this extra volatility value and hence so should the gap between SPIKES and VIX. The magnitude of the rise in this gap is a subject best left for future research.

Tables with all outcomes are summed up here.

Table 5: Calls and Puts contribution to $SPIKES^2$-VIX^2: no dividend paid during SPY options lifetime.

Interest Rate	Dividend Rate	Call Premium	Put Premium
0.01	0.01	0.001454794	-0.000805239
0.01	0.03	0.001290196	-0.000662787
0.01	0.05	0.001114029	-0.000530795
0.01	0.1	0.000620801	-0.000243255
0.03	0.01	0.001452367	-0.000803898
0.03	0.03	0.001288041	-0.000661683
0.03	0.05	0.001112182	-0.000529911
0.03	0.1	0.000619764	-0.00024285
0.05	0.01	0.001449951	-0.000802559
0.05	0.03	0.0012859	-0.000660581
0.05	0.05	0.001110329	-0.000529028
0.05	0.1	0.000618735	-0.000242445
0.1	0.01	0.001443918	-0.000799222
0.1	0.03	0.001280548	-0.000657834
0.1	0.05	0.001105702	-0.000526829
0.1	0.1	0.000616152	-0.000241437

References

[1] Black, F., and M. Scholes, 1973, The Pricing of Options and Corporate Liabilities, *Journal of Political Economy*, 81, 637âĂŞ-659.

[2] Carr, P., R. Jarrow, and R.Myneni, 1992, Alternative Characterizations of the American Put, *Mathematical Finance*, 2, 87-âĂŞ106.

[3] Chen, X., and J. Chadam, 2006, A Mathematical Analysis of the Optimal Exercise Boundary for American Put Options, *SIAM Journal of Mathematical Analysis*, 38, 1613âĂŞ-1641.

Table 6: Calls and Puts contribution to $sPIKES^2$-VIX^2: a single dividend is paid during SPY options lifetime in $T_d = 1$ week.

Interest Rate	Dividend Rate	Call Premium	Put Premium
0.01	0.01	-7.80E-05	8.88E-05
0.01	0.03	-0.000235692	0.000231281
0.01	0.05	-0.000381155	0.000363273
0.01	0.1	-0.000606554	0.00065081
0.03	0.01	-7.80E-05	8.88E-05
0.03	0.03	-0.000238416	0.000230991
0.03	0.05	-0.000390733	0.000362764
0.03	0.1	-0.000644965	0.000649832
0.05	0.01	-7.78E-05	8.91E-05
0.05	0.03	-0.000240016	0.000231057
0.05	0.05	-0.000398284	0.000362602
0.05	0.1	-0.00068005	0.000649191
0.1	0.01	-7.75E-05	9.32E-05
0.1	0.03	-0.00024079	0.000234633
0.1	0.05	-0.000409603	0.000365643
0.1	0.1	-0.000754455	0.00065104

Table 7: Calls and Puts contribution to $SPIKES^2$-VIX^2: a single dividend is paid during SPY options lifetime in $T_d = 2$ weeks.

Interest Rate	Dividend Rate	Call Premium	Put Premium
0.01	0.01	-7.33E-05	8.90E-05
0.01	0.03	-0.000189698	0.000231473
0.01	0.05	-0.000265527	0.000363469
0.01	0.1	-0.000275561	0.000651003
0.03	0.01	-7.60E-05	9.04E-05
0.03	0.03	-0.000200381	0.000232603
0.03	0.05	-0.000284308	0.000364381
0.03	0.1	-0.000315185	0.000651436
0.05	0.01	-7.74E-05	9.32E-05
0.05	0.03	-0.00020971	0.000235215
0.05	0.05	-0.000301732	0.000366779
0.05	0.1	-0.000353347	0.000653359
0.1	0.01	-7.75E-05	0.000106733
0.1	0.03	-0.000227261	0.000248123
0.1	0.05	-0.000339285	0.00037913
0.1	0.1	-0.000442443	0.000664521

Table 8: Calls and Puts contribution to $SPIKES^2$-VIX^2: a single dividend is paid during SPY options lifetime in $T_d = 3$ weeks.

Interest Rate	Dividend Rate	Call Premium	Put Premium
0.01	0.01	-6.13E-05	8.95E-05
0.01	0.03	-0.000130176	0.000231913
0.01	0.05	-0.000150251	0.000363906
0.01	0.1	-2.26E-05	0.000651437
0.03	0.01	-6.69E-05	9.23E-05
0.03	0.03	-0.000143021	0.000234472
0.03	0.05	-0.000168961	0.000366243
0.03	0.1	-5.47E-05	0.000653305
0.05	0.01	-7.16E-05	9.66E-05
0.05	0.03	-0.000155144	0.000238554
0.05	0.05	-0.00018702	0.000370099
0.05	0.1	-8.63E-05	0.000656685
0.1	0.01	-7.75E-05	0.000112425
0.1	0.03	-0.000182191	0.000253819
0.1	0.05	-0.000229235	0.000384813
0.1	0.1	-0.000162252	0.0006702

Table 9: Difference between SPIKES and VIX vega by letting r, q and σ vary:a no-dividends during SPY options lifetime

VegaS-VegaV							
	r	q	sigma	vega Put part	sigma	vega Call Part	VegaTot
	0.01	0.01	0.1	0.000223395	0.1	0.008339202	0.00856
	0.01	0.01	0.2	0.000244711	0.2	-0.01035607	-0.01011
	0.01	0.01	0.3	0.000258638	0.3	-0.01763977	-0.01738
	0.01	0.01	0.4	0.000263185	0.4	0.009118929	0.00938
	0.01	0.01	0.5	0.000278533	0.5	-0.05392467	-0.05365
	0.01	0.01	0.6	0.000289901	0.6	0.040794324	0.04108
	0.01	0.01	0.7	0.000291038	0.7	0.048344667	0.04864
	0.01	0.03	0.1	0.000223821	0.1	0.004313266	0.00454
	0.01	0.03	0.2	0.000245564	0.2	-0.00420946	-0.00396
	0.01	0.03	0.3	0.000258638	0.3	0.025528576	0.02579
	0.01	0.03	0.4	0.000262617	0.4	0.003133386	0.0034
	0.01	0.03	0.5	0.00027967	0.5	-0.01680291	-0.01652
	0.01	0.03	0.6	0.000287628	0.6	-0.03879451	-0.03851
	0.01	0.03	0.7	0.000291038	0.7	0.004375966	0.00467
	0.01	0.05	0.1	0.000223537	0.1	-0.00084119	-0.00062
	0.01	0.05	0.2	0.000245279	0.2	0.028968646	0.02921
	0.01	0.05	0.3	0.000258638	0.3	-0.0066992	-0.00644
	0.01	0.05	0.4	0.000262048	0.4	-0.01316071	-0.0129
	0.01	0.05	0.5	0.00027967	0.5	0.037121972	0.0374
	0.01	0.05	0.6	0.000287628	0.6	-0.02643247	-0.02614
	0.01	0.05	0.7	0.000292175	0.7	-0.01393524	-0.01364
	0.01	0.1	0.1	0.000223679	0.1	0.006974911	0.0072
	0.01	0.1	0.2	0.000244711	0.2	0.012848943	0.01309
	0.01	0.1	0.3	0.000258638	0.3	-0.00991659	-0.00966
	0.01	0.1	0.4	0.000263185	0.4	-0.00109004	-0.00083
	0.01	0.1	0.5	0.000278533	0.5	0.056777013	0.05706
	0.01	0.1	0.6	0.000289901	0.6	-0.00036272	-7.3E-05
	0.01	0.1	0.7	0.000291038	0.7	0.032621228	0.03291

Table 10: Difference between SPIKES and VIX vega by letting r, q and σ vary: discrete dividends for SPY options paid at time $T_d = 1$ week.

VegaS-VegaV							
	r	q	sigma	vega Put part	sigma	vega Call Part	VegaTot
	0.03	0.01	0.1	0.000587406	0.1	-0.02009966	-0.01951
	0.03	0.01	0.2	0.000650857	0.2	-0.00341143	-0.00276
	0.03	0.01	0.3	0.000692637	0.3	-0.02601429	-0.02532
	0.03	0.01	0.4	0.000723333	0.4	-0.00027234	0.00045
	0.03	0.01	0.5	0.000749765	0.5	0.091610249	0.09236
	0.03	0.01	0.6	0.000771934	0.6	-0.03677792	-0.03601
	0.03	0.01	0.7	0.000794103	0.7	-0.01253206	-0.01174
	0.03	0.03	0.1	0.000587406	0.1	-0.00095276	-0.00037
	0.03	0.03	0.2	0.000650786	0.2	0.006415141	0.00707
	0.03	0.03	0.3	0.000692637	0.3	-0.00711683	-0.00642
	0.03	0.03	0.4	0.000723617	0.4	4.34E-12	0.00072
	0.03	0.03	0.5	0.000750049	0.5	0.017284986	0.01804
	0.03	0.03	0.6	0.000771649	0.6	0.038316718	0.03909
	0.03	0.03	0.7	0.000793534	0.7	0.041746516	0.04254
	0.03	0.05	0.1	0.000587406	0.1	0.006523424	0.00711
	0.03	0.05	0.2	0.000650715	0.2	-0.00206317	-0.00141
	0.03	0.05	0.3	0.000692495	0.3	0.019678134	0.02037
	0.03	0.05	0.4	0.000723617	0.4	0.065942521	0.06667
	0.03	0.05	0.5	0.000750049	0.5	-0.04449133	-0.04374
	0.03	0.05	0.6	0.000771649	0.6	-0.03723885	-0.03647
	0.03	0.05	0.7	0.000793534	0.7	0.005204544	0.006
	0.03	0.1	0.1	0.000587441	0.1	0.008550079	0.00914
	0.03	0.1	0.2	0.000650857	0.2	-0.06411739	-0.06347
	0.03	0.1	0.3	0.000692637	0.3	-0.17861807	-0.17793
	0.03	0.1	0.4	0.000723901	0.4	0.001161537	0.00189
	0.03	0.1	0.5	0.00074948	0.5	-0.0099781	-0.00923
	0.03	0.1	0.6	0.000771649	0.6	0.092116368	0.09289
	0.03	0.1	0.7	0.000793534	0.7	-0.0082245	-0.00743

Table 11: Difference between SPIKES and VIX vega by letting r, q and σ vary: discrete dividends for SPY options paid at time $T_d = 2$ weeks.

VegaS-VegaVix							
	r	q	sigma	vega Put part	sigma	vega Call Part	VegaTot
	0.05	0.01	0.1	0.000909779	0.1	0.017668533	0.01858
	0.05	0.01	0.2	0.001017924	0.2	-0.0079947	-0.00698
	0.05	0.01	0.3	0.001087059	0.3	0.001007085	0.00209
	0.05	0.01	0.4	0.001140421	0.4	-0.04206829	-0.04093
	0.05	0.01	0.5	0.001185327	0.5	0.021870742	0.02306
	0.05	0.01	0.6	0.001224691	0.6	-0.0067628	-0.00554
	0.05	0.01	0.7	0.001261071	0.7	-0.06167975	-0.06042
	0.05	0.03	0.1	0.000909814	0.1	0.013649521	0.01456
	0.05	0.03	0.2	0.001017924	0.2	0.035760074	0.03678
	0.05	0.03	0.3	0.001086988	0.3	0.000907543	0.00199
	0.05	0.03	0.4	0.001140421	0.4	0.059913666	0.06105
	0.05	0.03	0.5	0.001185327	0.5	0.00491936	0.0061
	0.05	0.03	0.6	0.001224265	0.6	-0.01013951	-0.00892
	0.05	0.03	0.7	0.001261071	0.7	-0.04417054	-0.04291
	0.05	0.05	0.1	0.000909814	0.1	-0.00911493	-0.00821
	0.05	0.05	0.2	0.001017888	0.2	0.009915428	0.01093
	0.05	0.05	0.3	0.001086988	0.3	-0.00299156	-0.0019
	0.05	0.05	0.4	0.001140421	0.4	0.018527191	0.01967
	0.05	0.05	0.5	0.001185327	0.5	0.018440806	0.01963
	0.05	0.05	0.6	0.001224407	0.6	0.03023588	0.03146
	0.05	0.05	0.7	0.001260787	0.7	0.040372154	0.04163
	0.05	0.1	0.1	0.000909779	0.1	-0.02637969	-0.02547
	0.05	0.1	0.2	0.001017852	0.2	0.005457235	0.00648
	0.05	0.1	0.3	0.001087059	0.3	-0.03697385	-0.03589
	0.05	0.1	0.4	0.001140421	0.4	-0.05144974	-0.05031
	0.05	0.1	0.5	0.001185327	0.5	-0.09880307	-0.09762
	0.05	0.1	0.6	0.001224265	0.6	-0.03510227	-0.03388
	0.05	0.1	0.7	0.001260787	0.7	-0.03487982	-0.03362

Table 12: Difference between SPIKES and VIX vega by letting r, q and σ vary: discrete dividends for SPY options paid at time $T_d = 3$ weeks.

VegaS-VegaV							
	r	q	sigma	vega Put part	sigma	vega Call Part	VegaTot
	0.1	0.01	0.1	0.001626885	0.1	0.003443527	0.00507
	0.1	0.01	0.2	0.001846345	0.2	0.02664271	0.02849
	0.1	0.01	0.3	0.001987033	0.3	-0.01574298	-0.01376
	0.1	0.01	0.4	0.00209571	0.4	-0.02369572	-0.0216
	0.1	0.01	0.5	0.002187264	0.5	0.157677241	0.15986
	0.1	0.01	0.6	0.002267768	0.6	0.043872779	0.04614
	0.1	0.01	0.7	0.002341451	0.7	-0.08648874	-0.08415
	0.1	0.03	0.1	0.001626885	0.1	0.022871856	0.0245
	0.1	0.03	0.2	0.001846345	0.2	0.025201291	0.02705
	0.1	0.03	0.3	0.001986997	0.3	0.002142051	0.00413
	0.1	0.03	0.4	0.002095781	0.4	0.015824855	0.01792
	0.1	0.03	0.5	0.002187264	0.5	-0.01504297	-0.01286
	0.1	0.03	0.6	0.002267768	0.6	-0.00967773	-0.00741
	0.1	0.03	0.7	0.002341665	0.7	0.098902291	0.10124
	0.1	0.05	0.1	0.001626868	0.1	-0.00215557	-0.00053
	0.1	0.05	0.2	0.001846345	0.2	-0.03411346	-0.03227
	0.1	0.05	0.3	0.001986997	0.3	-0.02315964	-0.02117
	0.1	0.05	0.4	0.002095675	0.4	0.028724081	0.03082
	0.1	0.05	0.5	0.002187193	0.5	0.022247784	0.02443
	0.1	0.05	0.6	0.002267768	0.6	0.015490623	0.01776
	0.1	0.05	0.7	0.002341451	0.7	0.039771438	0.04211
	0.1	0.1	0.1	0.001626876	0.1	-0.00039496	0.00123
	0.1	0.1	0.2	0.001846345	0.2	-0.08763561	-0.08579
	0.1	0.1	0.3	0.001987033	0.3	-0.01207034	-0.01008
	0.1	0.1	0.4	0.002095746	0.4	-0.09752849	-0.09543
	0.1	0.1	0.5	0.002187193	0.5	-0.03023322	-0.02805
	0.1	0.1	0.6	0.002267768	0.6	-0.04203318	-0.03977
	0.1	0.1	0.7	0.002341594	0.7	-0.04769483	-0.04535

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